THERMOELASTIC STRESSES IN A LONG CYLINDER FOR THE CASE OF TRANSFER OF HEAT BETWEEN THE SURFACE AND AN EXTERNAL MEDIUM WITH VARIABLE TEMPERATURE

P. Z. Livshits

UDC 539.30:536.248

We examine a three-dimensional thermoelasticity problem for an isotropic circular cylinder. For a specified discontinuous temperature field the solution is derived in a form effective for numerical calculations.

As is well known, the solution of a steady-state nonaxisymmetric thermoelasticity problem (without consideration of mass forces and in the absence of heat sources) reduces to the integration of the basic system of equations (1), i.e.,

$$\nabla^2\bar{u} + \frac{1}{1-2\nu}\,\mathrm{grad}\,\mathrm{div}\,\bar{u} = \frac{2(1+\nu)}{1-2\nu}\,\alpha_t\,\mathrm{grad}\,T, \quad \nabla^2 T = 0. \tag{1}$$

The sought solution satisfying (1) and set by the boundary conditions can be given in the form of the sum of the particular solutions for $\bar{u}^{(t)}$ and for the general solution of $\bar{u}^{(e)}$, chosen [1, 2] in the form

$$u_r^{(t)} = 0,\ u_\varphi^{(t)} = 0,\ u_z^{(t)} = 2(1+\nu)\,\alpha_t \int T(r,\ \varphi,\ z)\,dz \quad (\nabla^2 u_z^{(t)} = 0), \tag{2}$$

$$u_r^{(e)} = r\frac{\partial^2\chi_1}{\partial z^2} + \frac{\partial\chi_2}{\partial r} + \frac{1}{r}\,\frac{\partial\chi_3}{\partial\varphi}, \quad u_\varphi^{(e)} = \frac{1}{r}\,\frac{\partial\chi_2}{\partial\varphi} - \frac{\partial\chi_3}{\partial r},$$

$$u_z^{(e)} = -r\,\frac{\partial^2\chi_1}{\partial r\,\partial z} - 4(1-\nu)\frac{\partial\chi_1}{\partial z} + \frac{\partial\chi_2}{\partial z},$$

$$\nabla^2\chi_k = 0 \qquad (k = 1,\ 2,\ 3). \tag{3}$$

The components of the stress tensor σ_{ij} are determined [3] from the formulas of Hooke's law

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{1}{1-2\nu}\,[\nu\vartheta - (1+\nu)\,\alpha_t T]\,\delta_{ij}\right\} \qquad (i,\ j = r,\ \varphi,\ z). \tag{4}$$

Thus, according to (2), we find $\sigma_{ij}^{(t)}$, while in accordance with (3), (for T = 0) we also calculate $\sigma_{ij}^{(e)}$

$$\sigma_{rr}^{(t)} = \sigma_{\varphi\varphi}^{(t)} = -\sigma_{zz}^{(t)} = -\alpha_t ET,$$

$$\sigma_{rz}^{(t)} = \alpha_t E\frac{\partial}{\partial r}\int T\,dz, \quad \sigma_{\varphi z}^{(t)} = \frac{\alpha_t E}{r}\,\frac{\partial}{\partial\varphi}\int T\,dz. \tag{5}$$

Let us consider the case involving the distribution of the temperature T_m of the external medium, in which the heating conditions ($T_m > T$) at the side surface of the cylinder ($r = a$) have the form

$$\frac{1}{h_t}\,\frac{\partial T}{\partial r} + T = T_m, \quad T_m = \begin{cases} T_n\cos n\varphi & \text{when } |z| < c, \\ 0 & \text{when } |z| > c. \end{cases} \tag{6}$$

In dimensionless coordinates ρ and ζ the solution for the heat-conduction equation satisfying boundary conditions (6) can be written in the form of the Fourier integral

Chemopharmaceutical Institute, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 16, No. 4, pp. 687-693, April, 1969. Original article submitted June 11, 1968.

TABLE 1. Values of $\sigma_\varphi/[\alpha_t ET]\cos n\varphi$ at the Surface ($\rho = 1$) of the Cylinder in the Middle ($\zeta = 0$) of the Heating Segment for Various Conditions of Heat Transfer with the Ambient Medium

Bi	n	b=0,1	b=0,3	b=0,5	b=1,0
1	0	—0,046	—0,084	—0,082	—0,050
	1	—0,080	—0,090	—0,072	—0,030
	2	—0,072	—0,072	—0,048	—0,016
2	0	—0,094	—0,142	—0,122	—0,060
	1	—0,126	—0,134	—0,102	—0,040
	2	—0,136	—0,118	—0,080	—0,024
5	0	—0,220	—0,236	—0,180	—0,064
	1	—0,244	—0,214	—0,148	—0,046
	2	—0,246	—0,192	—0,118	—0,030
10	0	—0,458	—0,350	—0,228	—0,066
	1	—0,332	—0,260	—0,166	—0,046
	2	—0,342	—0,236	—0,138	—0,034
100	0	—0,644	—0,400	—0,240	—0,062
	1	—0,560	—0,314	—0,174	—0,046
	2	—0,616	—0,306	—0,160	—0,038

$$T(\rho, \varphi, \zeta) = T_n \frac{2}{\pi} \cos n\varphi \int_0^\infty \frac{I_n(\beta\rho)\sin\beta b\cos\beta\zeta}{\beta\Lambda_n(\beta)}\, d\beta,$$
$$\Lambda_n(\beta) = I_n(\beta) + \frac{\beta I_n'(\beta)}{\mathrm{Bi}}. \qquad (7)$$

For the solution of the equations of elasticity theory (i.e., (1) for T = 0) we choose the functions $\chi_k(\rho, \varphi, \zeta)$ in (3) as follows:

$$\begin{pmatrix}\chi_1, & \chi_2\\ & \chi_3\end{pmatrix} = \frac{2(1+\nu)}{\pi E} a^2 \begin{pmatrix}\cos\\ \sin\end{pmatrix} n\varphi \int_0^\infty \begin{pmatrix}A, & B\\ & C\end{pmatrix} I_n(\beta\rho) \frac{\sin\beta b\cos\beta\zeta}{\beta}\, d\beta, \qquad (8)$$

where $A(\beta)$, $B(\beta)$, and $C(\beta)$ are determined from the condition that there are no total ($\sigma^{(t)} + \sigma^{(e)}$) stresses σ_{rr}, σ_{rz}, $\sigma_{r\varphi}$ (subsequently denoted, respectively, as σ_ρ, $\tau_{\rho z}$, $\tau_{\rho\varphi}$) on the $\rho = 1$ side surface of the cylinder.

Satisfaction of these boundary conditions (in conjunction with (7) and (5)) leads to a system of linear equations, whose expanded matrix has the form ($\theta = \theta_n = [\alpha_t ET_n]$)

$$\left\|\begin{matrix} \beta^2[-(1-2\nu)I-\beta I'], & [(\beta^2+n^2)I-\beta I'], & n(\beta I'-I), & \dfrac{\theta I}{\Lambda(\beta)} \\ \left[\left(\beta^2+\dfrac{n^2}{2}\right)I+2(1-\nu)\beta I'\right], & -\beta I', & -\dfrac{n}{2}I, & -\dfrac{\theta I'}{\beta\Lambda(\beta)} \\ \dfrac{n\beta^2}{2}I, & n(I-\beta I'), & \beta I'-\left(\dfrac{\beta^2}{2}+n^2\right)I, & 0 \end{matrix}\right\| \qquad (9)$$

Considering (3)-(5), (7), and (8), and introducing the notations $F_N^{(t)}(\rho, \beta)$, $F_N^{(e)}(\rho, \beta)$ (N = 1, 2, . . . , 6)

$$F_1^{(t)} = F_2^{(t)} = -F_3^{(t)} = \frac{\beta\rho}{n} F_5^{(t)} = -\frac{I(\beta\rho)}{\beta\Lambda(\beta)},$$
$$F_4^{(t)} = \frac{I'(\beta\rho)}{\beta\Lambda(\beta)}, \qquad (10)$$

$$F_N^{(e)} = \left[IF_N^{(1)} - \frac{I'}{\beta}F_N^{(2)}\right]\frac{1}{\Lambda(\beta)}, \quad F_N = F_N^{(t)} + F_N^{(e)}, \qquad (11)$$

we find the solution for the thermoelasticity problem in the form of the following integrals, e.g., (N = 1 and N = 4):

$$\frac{\sigma_\rho}{\theta\cos n\varphi} = \frac{2}{\pi}\int_0^\infty F_1(\rho, \beta)\sin\beta b\cos\beta\zeta\, d\beta,$$

$$\frac{\tau_{\rho z}}{\theta\cos n\varphi} = \frac{2}{\pi}\int_0^\infty F_4(\rho, \beta)\sin\beta b\sin\beta\zeta\, d\beta. \tag{12}$$

In (11) the functions $F_N^{(m)}$ (m = 1, 2) have the form, e.g., for N = 1,

$$F_1^{(m)} = \Big\{A_m\beta^2[-(1-2\nu)I(\beta\rho) - \beta\rho I'(\beta\rho)] + B_m\Big[\Big(\beta + \frac{n^2}{\rho^2}\Big)I(\beta\rho) - \frac{\beta}{\rho}I'(\beta\rho)\Big]$$
$$+ C_m\Big[\frac{n}{\rho}\Big(\beta I'(\beta\rho) - \frac{I(\beta\rho)}{\rho}\Big)\Big]\Big\}\frac{1}{\beta\Delta(\beta)} \equiv \{\cdots\}\frac{1}{\beta\Delta(\beta)} = \frac{\beta^{n+1}\sigma_\rho^{(m)}(\rho, \beta)}{\Delta(\beta)}, \tag{13}$$

$A_m(\beta)$, $B_m(\beta)$, and $C_m(\beta)$ are algebraic complements of the elements in the m-th row of the determinant $\Delta(\beta) \equiv \Delta_n(\beta)$ of the system (see (9)).

Integrals such as (12) are calculated on the basis of the Cauchy theorem by summation of the residues of the functions $F_N(\rho, \beta)\exp(i\beta\lambda)(\lambda > 0)$ over all the poles β_s in the upper half plane β (N = 1, 2, 3, 6)

$$\frac{2}{\pi}\int_0^\infty F_N(\rho, \beta)\sin\beta\lambda\, d\beta = \mathrm{res}_N(0) + 2\sum_{s=1}^\infty\{[\mathrm{res}\, F_N(\rho, i\varkappa_s)]\exp(-\varkappa_s\lambda)$$
$$+ [\mathrm{res}\, F_N(\rho, i\alpha_s)]\exp(-\alpha_s\lambda) + 2\,\mathrm{Re}\,[\mathrm{res}\, F_N(\rho, \beta)\exp(i\gamma_s\lambda)]\exp(-\delta_s\lambda)\}, \tag{14}$$

$$\mathrm{res}\, F_N(\rho, \beta) = \lim_{\beta\to\beta_s}(\beta - \beta_s)F_N(\rho, \beta),\ \lambda = b + \zeta,\ |b - \zeta|, \tag{15}$$

$\beta_s \equiv \beta_{ns} = \pm i\varkappa_s$ are the roots of the equation $\Lambda_n(\beta) = 0$ (7); $\beta_s = \beta_{ns} = \pm i\alpha_s$, $\beta_s \equiv \beta_{ns} = \pm\gamma_s \pm i\delta_s$ are the roots of Eq. (16), i.e.,

$$\Delta_n(\beta) \equiv \beta^{n+1}\psi_n(\beta) = 0. \tag{16}$$

The expressions for the stresses in (12), according to formulas such as (14), can now be presented in the following manner:

$$\sigma_\rho = \theta\Omega_1^t\cos n\varphi,\quad \sigma_z = \sigma_z^0 + \theta\Omega_3^t\cos n\varphi,\quad \tau_{\varphi z} = \theta\Omega_5^t\sin n\varphi,$$
$$\sigma_\varphi = \theta\Omega_2^t\cos n\varphi,\quad \tau_{\rho z} = \theta\Omega_4^t\cos n\varphi,\quad \tau_{\rho\varphi} = \theta\Omega_6^t\sin n\varphi, \tag{17}$$

where the quantity σ_z^0 (corresponding to the solution obtained as $\beta \to 0$ in the temperature distribution (7)) is equal to zero for n = 0 and n = 1,* while for n ≥ 2:

$$\frac{\sigma_z^0}{\cos n\varphi} = -\frac{\theta\rho^n j_0}{[1 + n(\mathrm{Bi})^{-1}]},\quad j_0 = \frac{2}{\pi}\int_0^\infty\frac{\sin\beta b\cos\beta\zeta}{\beta}d\beta = \begin{cases}1 & \zeta < b,\\ 1/2 & \zeta = b,\\ 0 & \zeta > b.\end{cases} \tag{18}$$

Thus if in this case $b \to \infty$ (i.e., the temperature T_m of the medium is constant over the entire length of the generatrix), formulas (17)† (for $\zeta > b$) in the case of n = 0 and n = 1 characterize the plane stressed state (when in an unattached solid cylinder, as is well known, no stresses arise), while for n ≥ 2 they characterize the state of plane deformation.

In (17) the term $\Omega_N^t(\rho, \zeta)$, which makes provision for the local nature of the heating, i.e., Eqs. (6), for N = 1, 2, 3, 6 is the following ($\zeta \geq 0$):

$$\Omega_N^\tau(\rho, \zeta) = \sum_{s=1}^\infty\{[\mathrm{res}\, F_N(\rho, i\varkappa_s)]\,[e^{-\varkappa_s(b+\zeta)} + \varepsilon e^{-\varkappa_s(|b-\zeta|)}]$$
$$+ [\mathrm{res}\, F_N(\rho, i\alpha_s)][e^{-\alpha_s(b+\zeta)} + \varepsilon e^{-\alpha_s(|b-\zeta|)}] + 2\,\mathrm{Re}\,[\mathrm{res}\, F_N(\rho, \beta_s)][e^{i\beta_s(b+\zeta)} + \varepsilon e^{i\beta_s(|b-\zeta|)}]\}. \tag{19}$$

*Since the cylinder is not attached, the pure bending stresses produced by a moment are eliminated from the consideration, beginning with (8) (for n = 1).

†We note that the axial force and the bending moment at any section ζ of the cylinder is equal to zero.

The multiplier ε in (19) assumes the following values:

$$\varepsilon = 1 \text{ when } \zeta < b, \ \varepsilon = 0 \text{ when } \zeta = b, \ \varepsilon = -1 \text{ when } \zeta > b, \tag{20}$$

while the quantities in the brackets, according to (15), (11), and (19), for example, when N = 1, have the form

$$[\operatorname{res} F_1(\rho, i\varkappa_s)] = \left\{-\frac{1}{\beta\Lambda'(\beta)}[I(\beta\rho) - \beta I(\beta) F_1^{(1)} + I'(\beta) F_1^{(2)}]\right\}_{\beta=i\varkappa_s}, \tag{21}$$

$$[\operatorname{res} F_1(\rho, i\alpha_s)] = \left[H(i\alpha_s)\frac{\sigma_\rho(\rho, i\alpha_s)}{\psi'(i\alpha_s)}\right],$$

$$[\operatorname{res} F_1(\rho, \beta_s)] = \left[H(\beta_s)\frac{\sigma_\rho(\rho, \beta_s)}{\psi'(\beta_s)}\right]. \tag{22}$$

In (22), on the basis of (15), (16), (13), and (11), we have used the notation $\sigma_\rho^{(1)}(\rho, i\alpha_s) = \sigma_\rho(\rho, i\alpha_s)$, $\sigma_\rho^{(1)}(\rho, \beta_s) = \sigma_\rho(\rho, \beta_s)$, . . . , and

$$H(\beta) = [I(\beta) + iI'(\beta) h(\beta)]\frac{1}{\Lambda(\beta)} \text{ when } \beta = i\alpha_s, \ \beta = \beta_s = \gamma_s + i\delta_s, \tag{23}$$

which accounts for the proportionality of the algebraic complements of the line elements of the determinant $\Delta(\beta)$ in (16) with the following values for its roots:

$$\frac{A_2(\beta)}{A_1(\beta)} = \frac{B_2(\beta)}{B_1(\beta)} = \frac{C_2(\beta)}{C_1(\beta)} = \frac{\beta}{i} h(\beta) \text{ when } \beta = i\alpha_s, \ \beta = \beta_s = \gamma_s + i\delta_s. \tag{24}$$

For N = 4.5 we have $\varepsilon \equiv -1$ in (19), while quantities similar to (21) and (22) should be replaced by corresponding terms, multiplied by $(-i)$, so that, for example,

$$[(-i)\operatorname{res} F_N(\rho, i\varkappa_s)] = (-i)\{\cdots\}_{\beta=i\varkappa_s},$$

$$[(-i)\operatorname{res} F_4(\rho, i\alpha_s)] = \left[H(i\alpha_s)\frac{\tau_{\rho z}(\rho, i\alpha_s)}{\psi'(i\alpha_s)}\right]. \tag{25}$$

As an example of the calculation, let us calculate the values of the stresses σ_φ and σ_z which arise in the middle (z = 0) of the (−c, c) segment of the side surface of a cylinder that is free of constraints and is heated by the medium: when $r = a$

$$\frac{\partial T}{\partial r} = h_t(T_m - T), \quad T_m = \begin{cases} T_0 + T_1\cos\varphi, & |z| < c, \\ 0 & |z| > c, \end{cases} \tag{26}$$

if we assume the length of the heating segment $2c = 0.5a$, and that the value of the Biot number is given by $Bi = ah_t = 2$.

On the strength of (17), (19), and (20), for the circumferential stresses σ_φ (N = 2) when $\rho = 1$ and $\zeta = 0$

$$\frac{\sigma_\varphi}{2\theta_n\cos n\varphi} = \sum_{s=1}^{\infty}\{[r_2(1, i\varkappa_s)]\exp(-\varkappa_s b) + [r_2(1, i\alpha_s)]\exp(-\alpha_s b) + 2\operatorname{Re}[r_2(1, \beta_s)\exp(i\beta_s b)]\}, \tag{27}$$

with the values of the roots $\varkappa_{0s}$ and β_{0s},* α_{1s}, β_{1s} needed for the calculations (with $\nu = 0.25$) contained, respectively, in [4, 5, 6], while the roots of $\varkappa_{1s}$ and the multipliers $[r_2(1, \beta)]$ for $\beta = i\varkappa_s$, $i\alpha_s$, and β_s have been tabulated by us for various values of Bi.

To illustrate the nature of the convergence of the residue series (27), we present the terms retained in the summation (for n = 0 and n = 1 where it is specified that b = 0.25):

$$[2\theta_0]^{-1}\sigma_\varphi = \{[-0.082 + 0 + 2(-0.011)] + [0.027 + 0 + 0] + [0.005 + 0 + 0]\} = -0.072,$$

$$[2\theta_1\cos\varphi]^{-1}\sigma_\varphi = \{[-1.645 + 1.560 + 2(-0.001)] + [0.012 + 0.001] + [0.003]\} = -0.071.$$

The axial stresses σ_z (N = 3) are calculated in similar fashion:

$$[2\theta_0]^{-1}\sigma_z = 0.034, \quad [2\theta_1\cos\varphi]^{-1}\sigma_z = 0.031.$$

---

*Here, when n = 0, there is no twisting, since the problem of twisting (with the characteristic roots of α_{0s}) is completely distinct from the problem under consideration.

When $T_1 = T_0$, we finally obtain in (26)

$$\sigma_\varphi = \theta_0[-0.144 - 0.142\cos\varphi],\quad \sigma_z = \theta_0[0.068 + 0.062\cos\varphi].$$

Thus, in the analyzed case of local heating (26), extremely substantial compressive stresses σ_φ are developed at the surface of the unattached solid cylinder, and there are tensile stresses σ_z approximately half as great (when the heating is uniform over the entire length of the cylinder generatrix, these stresses are equal to zero).

Table 1 gives the values of the stresses σ_φ, arising about the perimeter of the lateral cross section z = 0 of the cylinder in the case of local heating (6) when the temperature of the medium is constant about the perimeter (n = 0) and when it is variable (n = 1) and n = 2). In the calculations involving (27) we considered various combinations of the relative heating-segment length (2b) and the surface heat-transfer conditions (the values of the Bi number), and for n = 2 we used the values found (when $\nu = 0.25$) for the roots of the equation $\Delta_2(\beta) = 0$ (16):

$$\alpha_1 = 4.089,\ \alpha_2 = 8{,}114,\ \alpha_3 = 11.425;$$
$$\beta_1 = 0{,}959 + 2.104i,\ \beta_2 = 1.613 + 5.681i,\ \beta_3 = 1.842 + 9{,}033i.$$

From solution (17) we turn to the solution for the case of concentrated heating about the circumference (at the section $\zeta = 0$):

$$\text{when } \rho = 1 \ \lim 2bT_n = T_n^* a^{-1} \quad (2b \to 0,\ T_n \to \infty), \tag{28}$$

where T_n^* is the temperature per unit length of cylinder circumference.

The limit passage in (19) (for $\zeta > b$) leads to the following expressions for the functions $\omega_N^T(\rho, \zeta)$ for N = 1, 2, 3, 6:

$$\omega_N^t(\rho, \zeta) = \sum_{s=1}^{\infty} \{[-\varkappa_s \operatorname{res} F_N(\rho, i\varkappa_s)] \exp(-\varkappa_s\zeta) + $$
$$+ [-\alpha_s \operatorname{res} F_N(\rho, i\alpha_s)] \exp(-\alpha_s\zeta) + 2\operatorname{Re}[i\beta_s \operatorname{res} F_N(\rho, \beta_s)\exp(i\beta_s\zeta)]\}, \tag{29}$$

while for N = 4.5 they are similar in form (see (21)-(22) and (25)).

The expressions for the stresses in the case of concentrated (28) heating (6) can now be written, for example, as ($\theta^* \equiv \theta_n^* = [\alpha_t E T_n^*]$):

$$\frac{a\sigma_\rho}{\theta^*} = \omega_1^t \cos n\varphi,\quad \frac{a\sigma_z}{\theta^*} = \omega_3^t \cos n\varphi,\quad \frac{a\tau_{\varphi z}}{\theta^*} = \omega_5^t \sin n\varphi. \tag{30}$$

On the basis of the principle of superposition, we can extend these results to the case in which a segment of the side surface of the cylinder is heated in accordance with any law such that $T_m = T(a, \varphi, z)$.

NOTATION

$\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\varphi^2 + \partial^2/\partial z^2$	is the Laplace operator in a cylindrical coordinate system r, φ, z;
$\bar{u}$	is the displacement vector;
$T = T(r, \varphi, z)$	is the temperature at a point on the elastic cylinder;
ν	is the Poisson coefficient;
α_t	is the coefficient of linear thermal expansion;
E	is the modulus of elasticity;
$\vartheta = \partial u_r/\partial r + u_r/r + (1/r)\partial u_\varphi/\partial\varphi + \partial u_z/\partial z$	is the volume expansion;
ε_{ij}	is a component of the strain tensor;
δ_{ij}	is the Kronecker delta;
a	is the cylinder radius;
$\rho = r/a$ and $\zeta = z/a$	are dimensionless coordinates;
$b = c/a$	is the relative half-length of the heating segment;
h_t	is the relative heat-transfer coefficient;
$ah_t = Bi$	is the Biot number;
$I_n(x) \equiv I$ and $I_n'(x) \equiv I'$	are modified Bessel functions of n-th order and their derivatives with respect to the argument.

LITERATURE CITED

1. R. Shail, Intern. Engng. Sci., 3, No. 6, 597-602 (1965).
2. I. Dougall, Trans. Roy. Soc. Edinburgh, 49, p. IV, 895-978 (1914).
3. E. Melan and G. Parkus, Thermoelastic Stresses Produced by Steady-State Temperature Fields [Russian translation], GIFML (1958).
4. A. V. Luikov, The Theory of Heat Conduction [in Russian], Vysshaya Shkola (1967).
5. A. I. Lur'e, Three-Dimensional Problems in Elasticity Theory [in Russian], GITTL (1955).
6. P. Z. Livshits, Inzh. Zhurnal MTT, No. 4, 142-150 (1966).